SKEW-NORMAL CORRECTION TO GEODETIC DIRECTIONS ON AN ELLIPSOID

In Figure 1, P_1 and P_2 are two points at heights h_1 and h_2 above an ellipsoid of semi-major axis *a,* and flattening *f.* The normals P_1H_1 and P_2H_2 (piercing the ellipsoid at Q_1 and Q_2) are *skewed* with respect to each other. An observer at P_1, whose theodolite is set up so that its axis of revolution is coincident with the normal at P_1, sights to a target P_2; the vertical plane of the theodolite containing P_1, P_2 and H_1 is a normal section plane and will intersect the ellipsoid along the normal section curve Q_1Q_2' having an azimuth α_{12}' – <u>but this is not the correct normal section curve</u>.

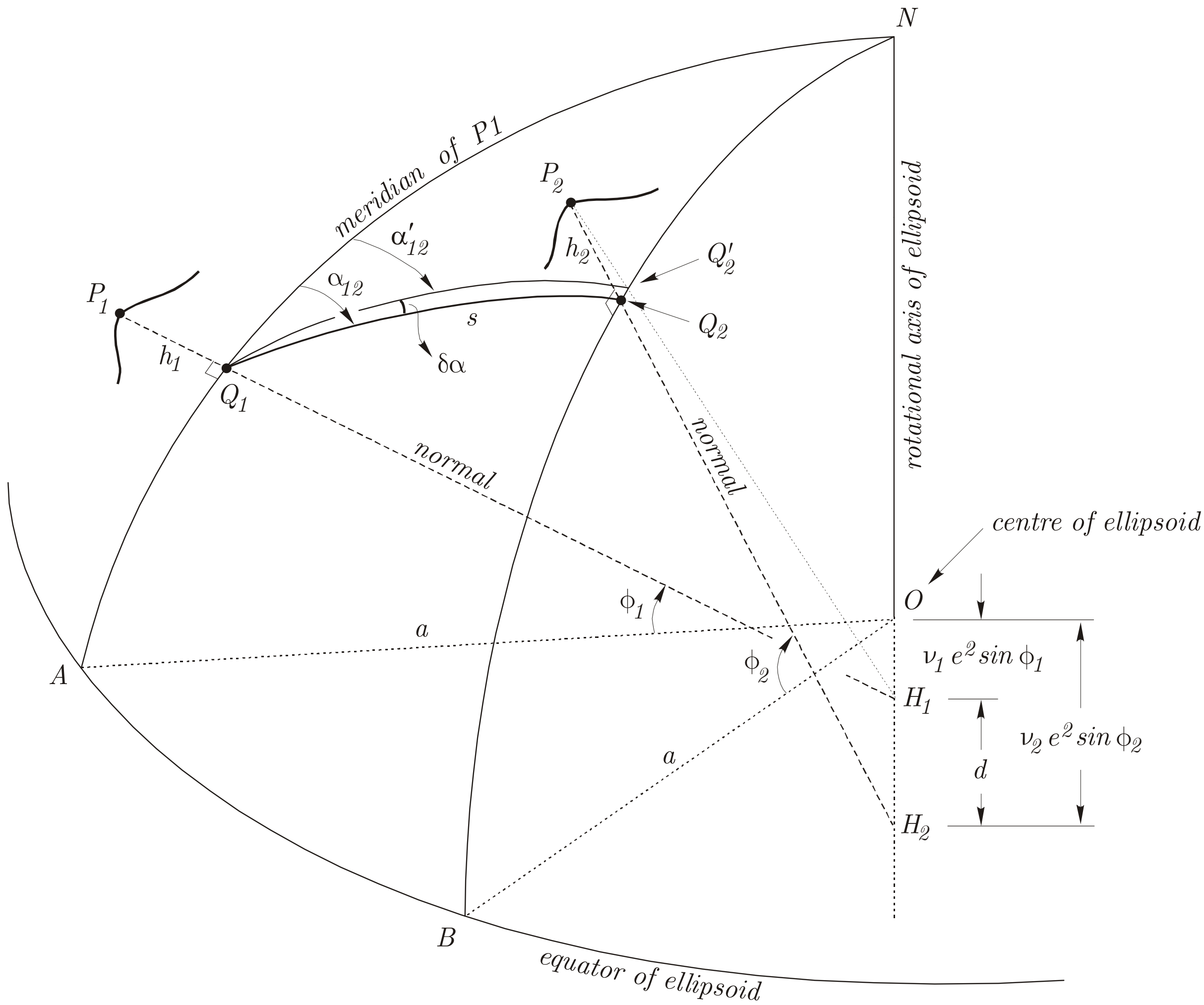


Figure 1. Sectional view of points P_1 at height h_1 and P_2 at height h_2 above an ellipsoid of revolution.

The correct normal section curve is the plane curve Q_1Q_2 having an azimuth α_{12} that is created by the intersection of the normal section plane containing P_1, Q_2 and H_1, and the ellipsoid surface. The angular difference between these two curves is $\delta\alpha$ and is applied as a correction to any observed direction to a target above the ellipsoid. This correction, known as the *skew-normal correction,* is due entirely to the height of the target station. It is sometimes known as the "height of target" correction and a formula for this correction is derived in the following manner.

From Figure 1

$$\begin{aligned} d &= \nu_2 e^2 \sin\phi_2 - \nu_1 e^2 \sin\phi_1 \\ &= ae^2\left(\frac{\nu_2}{a}\sin\phi_2 - \frac{\nu_1}{a}\sin\phi_1\right) \end{aligned} \tag{1}$$

d is the distance between H_1 and H_2 where the normals intersect the axis of revolution of the ellipsoid, ϕ is the latitude of P, $\nu = QH = a\big/\sqrt{1-e^2\sin^2\phi}$ is the radius of curvature of the prime vertical section of the ellipsoid at Q and $e^2 = f(2-f)$ is the square of the eccentricity of the ellipsoid.

In equation (1) the term ν/a will be very close to unity and we may write

$$\frac{\nu_1}{a} = 1+\varepsilon_1 \quad \text{and} \quad \frac{\nu_2}{a} = 1+\varepsilon_2$$

where ε are small positive quantities, and

$$\begin{aligned} d &= ae^2\left\{(1+\varepsilon_2)\sin\phi_2 - (1+\varepsilon_1)\sin\phi_1\right\} \\ &= ae^2\left\{(\sin\phi_2 - \sin\phi_1) + (\varepsilon_2\sin\phi_2 - \varepsilon_1\sin\phi_1)\right\} \end{aligned}$$

The 2nd term in the braces will be quite small, since ε_1 and ε_2 will be approximately 0.002 for mid-latitude values and may be neglected in an approximation, giving

$$d \cong ae^2(\sin\phi_2 - \sin\phi_1) \tag{2}$$

Using the trigonometric addition formula: $\sin A - \sin B = 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$

$$d = ae^2\left\{2\cos\left(\frac{\phi_1+\phi_2}{2}\right)\sin\left(\frac{\phi_2-\phi_1}{2}\right)\right\}$$

Letting the mean latitude $\phi_m = \dfrac{\phi_1 + \phi_2}{2}$ and the latitude difference $\delta\phi = \phi_2 - \phi_1$ gives

$$d = ae^2\left(2\cos\phi_m \sin\frac{\delta\phi}{2}\right)$$

Now if $\delta\phi$ is small then $\sin\dfrac{\delta\phi}{2} \cong \dfrac{\delta\phi}{2}$ and

$$\begin{aligned} d &\cong ae^2\left\{2\cos\phi_m\left(\frac{\delta\phi}{2}\right)\right\} \\ &= ae^2\left(\phi_2 - \phi_1\right)\cos\phi_m \end{aligned} \tag{3}$$

On the ellipsoid, the elemental distance along the meridian is $\rho\, d\phi$ where

$\rho = \dfrac{a\left(1-e^2\right)}{\left(1-e^2\sin^2\phi\right)^{3/2}}$ is the radius of curvature in the meridian plane and from Figure 2

$\rho\, d\phi = ds\cos\alpha$

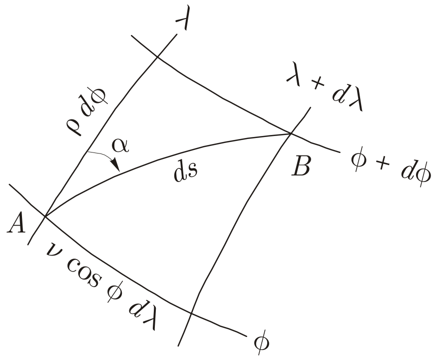


Figure 2. Elemental rectangle on the ellipsoid

Letting $\rho_m = \dfrac{\rho_1 + \rho_2}{2}$ we may write for a small rectangle on the ellipsoid

$$s\cos\alpha_{12} \cong \rho_m\,\delta\phi$$

and rearranging gives

$$\delta\phi = \phi_2 - \phi_1 \cong \frac{s\cos\alpha_{12}}{\rho_m} \tag{4}$$

Substituting equation (4) in equation (3) gives

$$d \cong ae^2\frac{s}{\rho_m}\cos\alpha_{12}\cos\phi_m \tag{5}$$

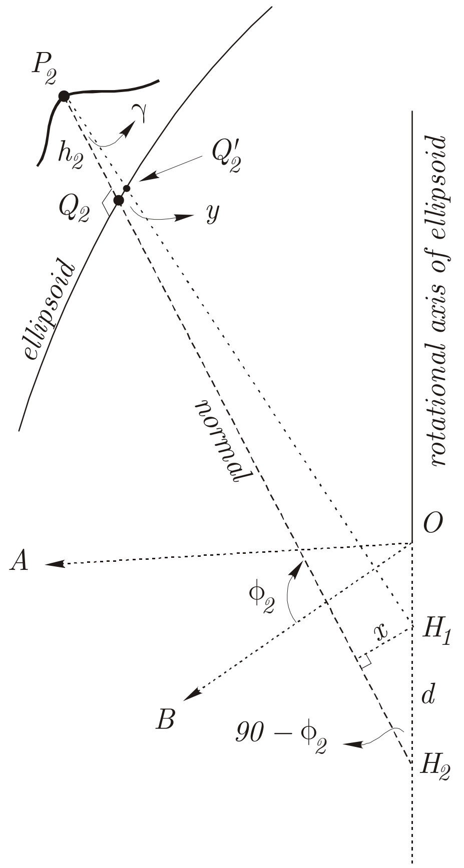


Figure 3 is extracted from Figure 1 and shows a schematic view of the meridian section through P_2.

The meridian arc distance $Q_2 Q_2' = y$

γ is the angle between the normal P_2H_2 and the line P_2H_1

x is the perpendicular distance from the normal to H_1 and

$$x = d\cos\phi_2 \tag{6}$$

Figure 3. Meridian section through P_2

Replacing ϕ_m with ϕ_2 in equation (5) since it will not introduce any appreciable error gives

$$d \cong ae^2 \frac{s}{\rho_m}\cos\alpha_{12}\cos\phi_2$$

Multiplying both sides by $\cos\phi_2$ gives

$$d\cos\phi_2 = ae^2 \frac{s}{\rho_m}\cos\alpha_{12}\cos^2\phi_2$$

and the left-hand-side equals x of equation (6) hence

$$x = ae^2 \frac{s}{\rho_m}\cos\alpha_{12}\cos^2\phi_2 \tag{7}$$

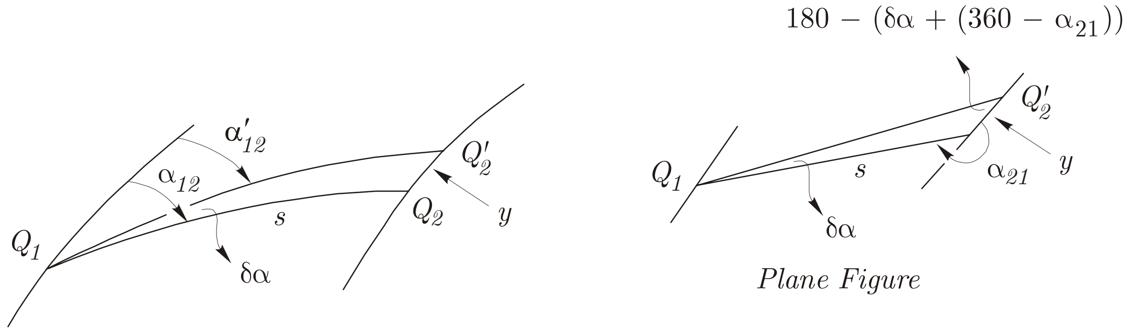
Referring to Figure 3, the distance P_2H_1 can be approximated by the semi-major axis length a, and the angle γ at P_2 approximated by $\gamma \cong \dfrac{x}{a}$; hence dividing both sides of equation (7) by a gives an approximation for the angle γ

$$\gamma \cong e^2 \frac{s}{\rho_m} \cos\alpha_{12} \cos^2\phi_2 \tag{8}$$

Now the distance $Q_2 Q_2' = y$ is approximately $h_2\gamma$ and from equation (8)

$$y \cong h_2 e^2 \frac{s}{\rho_m} \cos\alpha_{12} \cos^2\phi_2 \tag{9}$$

Referring to Figure 1, the spheroidal triangle $Q_1 Q_2 Q_2'$ can be considered plane since the angle $\delta\alpha$ is very small



In the plane figure $Q_1 Q_2 Q_2'$ the distance $Q_2 Q_2' = y$, the angle at Q_2' can be approximated as $\alpha_{21} - 180$ since $\delta\alpha$ is very small and the *sine rule* gives

$$\frac{s}{\sin\left(\alpha_{21} - 180\right)} = \frac{y}{\sin\delta\alpha}$$

Since $\delta\alpha$ is small $\sin\delta\alpha \cong \delta\alpha$ and $\alpha_{21} - 180 \cong \alpha_{12}$ we may write

$$\delta\alpha \cong \frac{y}{s}\sin\alpha_{12}$$

Substituting equation (9) into this equation gives the *skew-normal correction*

$$\boxed{\delta\alpha \cong \frac{h_2}{\rho_m} e^2 \sin\alpha_{12} \cos\alpha_{12} \cos^2\phi_2} \tag{10}$$

Using the trigonometric double angle formula $\sin 2A = 2\sin A\cos A$ gives an alternative expression

$$\delta\alpha \cong \frac{h_2}{2\rho_m} e^2 \sin 2\alpha_{12} \cos^2\phi_2 \tag{11}$$

The correction is applied in the following manner

$$\boxed{\text{correct normal section} = \text{observed normal section} + \delta\alpha}$$

To test the validity of formula for the skew-normal correction, equations (10) or (11), a series of test lines (geodesics) of 10, 20, 50, 100 and 200 km lengths with geodesic azimuths of 45° were computed. These lines of varying length radiate from P1 ($\phi = -38^\circ$, $\lambda = 145^\circ$) to points P2, P3, P4, P5 and P6. All the points are related to the GRS80 ellipsoid ($a = 6378137$ m, $f = 1/298.257222101$). Note that an azimuth of 45° will give the maximum value of $\sin\alpha_{12}\cos\alpha_{12}$ in equation (10). Table 1 shows the computed latitudes and longitudes of the terminal points.

Point	Geodesic Azimuth A_{12}	Geodesic distance s	Latitude	Longitude
P1			-38° 00′ 00″	145° 00′ 00″
P2	45° 00′ 00″	10,000.000	-37° 56′ 10.5605″	145° 04′ 49.5723″
P3	45° 00′ 00″	20,000.000	-37° 52′ 20.9209″	145° 09′ 38.6447″
P4	45° 00′ 00″	50,000.000	-37° 40′ 50.8093″	145° 24′ 02.8787″
P5	45° 00′ 00″	100,000.000	-37° 21′ 36.6945″	145° 47′ 53.4183″
P6	45° 00′ 00″	200,000.000	-36° 42′ 54.0745″	146° 34′ 58.2597″

Table 1 Test Lines P1-P2, P1-P3, …, P1-P6 on the GRS80 ellipsoid

<u>Assuming ellipsoidal heights $h = 0$</u> the "correct" normal section azimuths α_{12} can be computed from Cartesian coordinate differences. Table 2 shows the "correct" normal section azimuths of the test lines P1-P2, P1-P3, …, P1-P6.

Point	Normal section azimuth α_{12}	Radius of curvature of prime vertical section (nu)	Cartesian coordinates		
			X	Y	Z
P1		6386244.475125	-4122324.7665	2886482.8764	-3905443.9683
P2	45° 00′ 00.0148″	6386221.351640	-4129941.5802	2883184.0499	-3899867.0633
P3	45° 00′ 00.0054″	6386198.221201	-4137548.2456	2879878.1402	-3894280.5111
P4	45° 00′ 00.0052″	6386128.790435	-4160307.1712	2869917.9837	-3877463.1076
P5	45° 00′ 00.0207″	6386012.954750	-4198034.0032	2853176.7895	-3849242.4481
P6	45° 00′ 00.0721″	6385780.944705	-4272711.6267	2819169.5729	-3792088.6502

Table 2 Normal section azimuths α_{12} of Test Lines on the GRS80 ellipsoid

Changing the ellipsoidal heights of stations P2, P3, P4, P5 and P6 to $h = 1000\,\mathrm{m}$, recomputing the Cartesian coordinates and then computing another set of "observed" normal section azimuths gives the values α'_{12} for $h = 1000\,\mathrm{m}$. These can then be compared with the "correct" azimuths α_{12} to obtain an exact value of the skew-normal correction $\delta\alpha$. This value can then be compared with the value computed from equation (10) to gauge the accuracy of the formula.

1	2	3	4	5	6
Point	Normal section azimuth α_{12}	α'_{12} for h = 1000m	true δa $\delta\alpha = \alpha_{12} - \alpha'_{12}$	computed δa eq (10)	diff
P1					
P2	45° 00′ 00.0148″	44° 59′ 59.9471″	0.0677″	0.0675″	+0.0002″
P3	45° 00′ 00.0054″	44° 59′ 59.9377″	0.0677″	0.0676″	+0.0001″
P4	45° 00′ 00.0052″	44° 59′ 59.9373″	0.0679″	0.0680″	-0.0001″
P5	45° 00′ 00.0207″	44° 59′ 59.9523″	0.0684″	0.0686″	-0.0002″
P6	45° 00′ 00.0721″	45° 00′ 00.0033″	0.0688″	0.0698″	-0.0010″

Table 3 Normal section azimuth corrections of Test Lines on the GRS80 ellipsoid

In Table 3, column 2 shows the "correct" normal section azimuths and column 3 shows the "observed" normal section azimuths for targets 1000 m above the ellipsoid. Column 4 shows the true correction and column 5 shows the correction computed from equation (10). The differences between the two corrections are shown in column 5 and from these we can infer that the correction as computed from equation (10) is accurate to at least 0.001″ for lines up to 100 km in length for targets 1000 m above the ellipsoid.

The values in Table 3 are "maximum" values for targets 1000 m above the ellipsoid in latitudes 37°–38° south. Inspection of equation (10) shows that the correction is proportional to the height of the target, so maximum values for targets 500 m above the ellipsoid in the same latitudes would be approximately 0.033″.

This correction is very small and is often ignored unless the terrain is mountainous.

REFERENCES

Krakiwsky, E.J. and Thomson, D.B., 1974. *Geodetic Position Computations,* 1990 re-print, Department of Surveying Engineering, University of Calgary, Calgary, Alberta, Canada.